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The Microscopic Stress Tensor Field in Particle Systems with Many-Body Interactions

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It is argued that the up to now only existing expression for the microscopic stress tensor in the presence of many-body interactions, while being formally acceptable, displays some physical shortcomings. These unpleasant features are remedied by explicitly constructing and symmetrizing a new stress tensor field. With the help of this construction, some recent results on the appearance of extremely long-ranged correlations involving the stress tensor field in systems with spontaneously broken symmetries are generalized.

KEY WORDS: Stress tensor; (angular) momentum conservation; inhomogeneous fluids; broken symmetries.

1. INTRODUCTION

The concept of the microscopic stress tensor field $\sigma^{\alpha\beta}(r)$ as originally introduced by Irving and Kirkwood[1] has its roots in the statistical mechanical theory of transport processes. It originates from the desire to construct a local quantity which describes the momentum flux in many-particle systems with translation-invariant interactions. In other words, one intends to fulfill a sort of continuity equation for the momentum density $p^\alpha(r)$ having the form

$$\frac{\partial}{\partial t} p^\alpha(r) = \sum_\beta \frac{\partial}{\partial x^\beta} \sigma^{\alpha\beta}(r) \tag{1.1}$$

A detailed discussion of this topic has been given in an excellent paper by Schofield and Henderson.[2] Some recent results on stress tensors are

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contained in three papers by Requardt and Wagner,[3-5] where it is shown that such tensor fields play a key role for the implications of all sorts of spontaneous symmetry breakdown. In particular, it is the appearance of extremely long-ranged correlations involving the stress tensor field that signals the existence of density inhomogeneities which are not induced by exterior fields.

The way to construct the stress tensor field $\sigma^{\alpha\beta}(r)$ is well established for the case of translation-invariant pair interactions with Hamiltonian

$$H=\sum_i p_i^2/2m+(1/2)\sum_{i\neq j} V(r_i-r_j) \tag{1.2}$$

The expression for $\sigma^{\alpha\beta}$ is of course not unique, since (1.1) fixes it only up to a tensor field $\delta\sigma^{\alpha\beta}$ with

$$\sum_\beta \frac{\partial}{\partial x^\beta}\,\delta\sigma^{\alpha\beta}(r)\equiv 0 \tag{1.3}$$

The commonly used form for $\sigma^{\alpha\beta}$ is[2,3]

$$\begin{aligned}\sigma^{\alpha\beta}(r)= &-(1/m)\sum_i p_i^\alpha p_i^\beta\,\delta(r-r_i)\\ &+(1/2)\sum_{i\neq j}(x_i^\beta-x_j^\beta)\,\partial^\alpha V(r_i-r_j)\int_0^1 ds\,\delta(r-sr_i-(1-s)\,r_j)\end{aligned} \tag{1.4}$$

(I often write ∂^α instead of $\partial/\partial x^\alpha$).

If the pair interaction is also rotation invariant, i.e., $V(r_i-r_j)=V(|r_i-r_j|)$, the above stress tensor is symmetric ($\sigma^{\alpha\beta}=\sigma^{\beta\alpha}$) and this leads to an additional continuity equation involving the angular momentum density $l^{\alpha\beta}(r)=x^\alpha p^\beta(r)-x^\beta p^\alpha(r)$ due to

$$\begin{aligned}\frac{\partial}{\partial t}\,l^{\alpha\beta}(r)&=\sum_\gamma\,[x^\alpha\partial^\gamma\sigma^{\beta\gamma}(r)-x^\beta\partial^\gamma\sigma^{\alpha\gamma}(r)]\\ &=\sigma^{\alpha\beta}(r)-\sigma^{\beta\alpha}(r)+\sum_\gamma\partial^\gamma[x^\alpha\sigma^{\beta\gamma}(r)-x^\beta\sigma^{\alpha\gamma}(r)]\end{aligned} \tag{1.5}$$

While the above results on systems with pair interactions are relatively widespread in the literature, knowledge about the form and the properties of the microscopic stress tensor in the presence of many-body potentials is surprisingly scarce. On the other hand, the treatment of such many-body effects is indispensible from a physical point of view, since it is already impossible to describe even the simplest classical fluids, such as, e.g., liquid argon, by using pair interactions alone.[6]

To the best of my knowledge, the only attempt to explicitly construct a microscopic stress tensor in the presence of multiparticle potentials has been carried through by Schofield and Henderson.[2] Taking

$$H = \sum_i p_i^2/2m + H_1(\{r_k\}) \tag{1.6}$$

with a general translation- and rotation-invariant interaction term $H_1(\{r_k\})$, they were led to the following result:

$$\begin{aligned}\sigma^{\alpha\beta}(r) = & -(1/m) \sum_i p_i^\alpha p_i^\beta \delta(r - r_i) \\ & + (1/4) \sum_{i,j} \sum_\gamma (x_i^\beta - x_j^\beta)[(x_i^\alpha - x_j^\alpha)\, \partial_i^\gamma \partial_j^\gamma - (x_i^\gamma - x_j^\gamma)\, \partial_i^\alpha \partial_j^\gamma \\ & + (1/2)(x_i^\gamma + x_j^\gamma)(\partial_i^\alpha \partial_j^\gamma - \partial_i^\gamma \partial_j^\alpha)]\, H_1(\{r_k\}) \\ & \times \int_0^1 ds\, \delta(r - sr_i - (1-s)\, r_j)\end{aligned} \tag{1.7}$$

At first glance, this expression looks relatively well-behaved, indeed fulfills the continuity equation (1.1), and is therefore on a formal level absolutely acceptable. However, I argue in the following that from a physical point of view the stress tensor field (1.7) displays two serious shortcomings. The first unpleasant feature to be discussed is the fact that the mere construction of the stress tensor (1.7) already requires both translation *and* rotation invariance of the interaction. This is at odds with what one expects on physical grounds, namely that translation invariance is already enough to imply momentum conservation as is reflected in the continuity equation (1.1). Rotation invariance should only be used to derive the additional conservation of angular momentum. This means that translation invariance alone should suffice to guarantee the existence of a well-defined stress tensor, whereas rotation invariance is only needed for the symmetrization of $\sigma^{\alpha\beta}$.

The second disadvantage is based on the fact that the stress tensor field as a physical observable should not contain contributions that are explicitly dependent on the origin of coordinates. This property is needed to exclude an influence of the choice of the coordinate system on the (decay) properties of correlation functions involving $\sigma^{\alpha\beta}$ (cf. in this respect the remarks on p. 499 of ref. 4). Therefore, the stress tensor field should have the following property[4]:

$$\sigma^{\alpha\beta}(r) = \sigma^{\alpha\beta}(r; \{r_k, p_k\}) = \sigma^{\alpha\beta}(r + a; \{r_k + a, p_k\}) \tag{1.8}$$

The stress tensor for pair interactions (1.4) is a case in point for this property, since coordinate differences of the form $r - r_i$, $r_i - r_j$, and $r - sr_i - (1 - s) r_j$ are clearly invariant under the substitution $(r; \{r_k\}) \to (r + a; \{r_k + a\})$.

On the other hand, expression (1.7) in general explicitly violates condition (1.8) due to the additional appearance of coordinate sums $x_i^\gamma + x_j^\gamma$. Therefore, this tensor field is influenced by coordinate system choices and is not a merely physical quantity. This means that the stress tensor (1.7) is in particular unsuitable for a generalization of the results of refs. 3–5 to systems with many-body interactions.

The preceding discussion has provided sufficient motivation to embark in the following sections on the construction and symmetrization of new well-defined stress tensor fields for multiparticle interactions which do not suffer from the diseases mentioned above.

2. CONSTRUCTION AND SYMMETRIZATION OF STRESS TENSOR FIELDS

In this section I want to show that it is indeed possible to construct a new well-behaved stress tensor field—in particular fulfilling (1.8)—whenever the many-body interactions are translation invariant and that it can be symmetrized if the interactions in addition show rotation invariance. To reach these aims, let us write the Hamiltonian (1.6) somewhat more explicitly:

$$H = \sum_i p_i^2/2m + \sum_l \frac{1}{l!} \sum_{i_1 \cdots i_l}^{\neq} V^{(l)}(r_{i_1}, \ldots, r_{i_l}) \tag{2.1}$$

($\sum^{\neq}$ denotes summation over pairwise different indices).

With the momentum density

$$p^\alpha(r) = \sum_i p_i^\alpha \delta(r - r_i) \tag{2.2}$$

we now calculate

$$\begin{aligned} \frac{\partial}{\partial t} p^\alpha(r) &= \{p^\alpha(r), H\} \\ &= -(1/m) \sum_{i,\beta} p_i^\alpha p_i^\beta \partial^\beta \delta(r - r_i) \\ &\quad - \sum_l \frac{1}{l!} \sum_{i_1 \cdots i_l}^{\neq} \sum_{k=1}^{l} \partial_{i_k}^\alpha V^{(l)}(r_{i_1}, \ldots, r_{i_l})\, \delta(r - r_{i_k}) \end{aligned} \tag{2.3}$$

The kinetic part of this expression is of course not affected by the introduction of multibody potentials and already has the desired form. For translation-invariant potentials $V^{(l)}$, i.e.,

$$\sum_{k=1}^{l} \partial_k^\alpha V^{(l)}(r_1,\ldots, r_l) \equiv 0 \tag{2.4}$$

one can rewrite the interaction part of (2.3) as follows:

$$\begin{aligned}
-\sum_l \frac{1}{l!} &\sum_{i_1\cdots i_l}^{\neq} \sum_{k=1}^{l} \partial_{i_k}^\alpha V^{(l)}(r_{i_1},\ldots, r_{i_l})\, \delta(r-r_{i_k}) \\
&= -\sum_l \frac{1}{l!} \sum_{i_1\cdots i_l}^{\neq} \frac{1}{l} \sum_{j,k=1}^{l} \partial_{i_k}^\alpha V^{(l)}(r_{i_1},\ldots, r_{i_l}) \\
&\quad\times [\delta(r-r_{i_k}) - \delta(r-r_{i_j})] \\
&= -\sum_l \frac{1}{l!} \sum_{i_1\cdots i_l}^{\neq} \frac{1}{l} \sum_{j\neq k} \partial_{i_k}^\alpha V^{(l)}(r_{i_1},\ldots, r_{i_l}) \\
&\quad\times \int_0^1 ds \frac{d}{ds}\, \delta(r - sr_{i_k} - (1-s)\, r_{i_j}) \\
&= \sum_l \frac{1}{l!} \sum_{i_1\cdots i_l}^{\neq} \frac{1}{l} \sum_{j\neq k} \partial_{i_k}^\alpha V^{(l)}(r_{i_1},\ldots, r_{i_l}) \\
&\quad\times \int_0^1 ds \sum_\beta (x_{i_k}^\beta - x_{i_j}^\beta)\, \partial^\beta \delta(r - sr_{i_k} - (1-s)\, r_{i_j})
\end{aligned} \tag{2.5}$$

Therefore we have constructed our stress tensor field

$$\begin{aligned}
\sigma^{\alpha\beta}(r) = &-(1/m) \sum_i p_i^\alpha p_i^\beta \delta(r-r_i) \\
&+ \sum_l \frac{1}{l!} \sum_{i_1\cdots i_l}^{\neq} \frac{1}{l} \sum_{j\neq k} (x_{i_k}^\beta - x_{i_j}^\beta) \\
&\times \partial_{i_k}^\alpha V^{(l)}(r_{i_1},\ldots, r_{i_l}) \int_0^1 ds\, \delta(r - sr_{i_k} - (1-s)\, r_{i_j})
\end{aligned} \tag{2.6}$$

where we indeed only needed translation invariance of the interaction. Moreover, (1.8) is fulfilled.

However, even if we in addition require rotation invariance of the potentials $V^{(l)}$, i.e.,

$$\sum_{k=1}^{l} (x_k^\alpha \partial_k^\beta - x_k^\beta \partial_k^\alpha)\, V^{(l)}(r_1,\ldots, r_l) \equiv 0 \tag{2.7}$$

the stress tensor (2.6) is in general not symmetric. To remove this nasty feature, we shall use the fact that the definition of the stress tensor is not unique since there is the possibility to add a correction term $\delta\sigma^{\alpha\beta}$ obeying (1.3).

General considerations due to McLennan[7] show us that it is always possible to symmetrize a nonsymmetric stress tensor by adding such a correction whenever

$$\sigma^{\alpha\beta}(r) - \sigma^{\beta\alpha}(r) = \sum_{\gamma} \partial^{\gamma} \eta^{\alpha\beta\gamma}(r) \tag{2.8}$$

with a sufficiently well-behaved third-rank tensor field $\eta^{\alpha\beta\gamma}(r)$. In particular, this tensor field has to satisfy the analog of (1.8). Indeed, if one defines

$$\begin{aligned}\delta\sigma^{\alpha\beta}(r) = &-\tfrac{1}{2}[\sigma^{\alpha\beta}(r) - \sigma^{\beta\alpha}(r)] \\ &- \tfrac{1}{4}\sum_{\gamma} \partial^{\gamma}[\eta^{\gamma\alpha\beta}(r) + \eta^{\gamma\beta\alpha}(r) - \eta^{\alpha\gamma\beta}(r) - \eta^{\beta\gamma\alpha}(r)]\end{aligned} \tag{2.9}$$

one immediately verifies (1.3) by using (2.8), and

$$\begin{aligned}\sigma_S^{\alpha\beta}(r) &= \sigma^{\alpha\beta}(r) + \delta\sigma^{\alpha\beta}(r) \\ &= \tfrac{1}{2}[\sigma^{\alpha\beta}(r) + \sigma^{\beta\alpha}(r)] \\ &\quad - \tfrac{1}{4}\sum_{\gamma} \partial^{\gamma}[\eta^{\gamma\alpha\beta}(r) + \eta^{\gamma\beta\alpha}(r) - \eta^{\alpha\gamma\beta}(r) - \eta^{\beta\gamma\alpha}(r)]\end{aligned} \tag{2.10}$$

is obviously symmetric.

So we have to check whether the antisymmetric part of the stress tensor field (2.6) indeed can be brought into line with (2.8) whenever the invariance properties (2.4) and (2.7) are valid. To this end we write

$$\begin{aligned}&\sigma^{\alpha\beta}(r) - \sigma^{\beta\alpha}(r) \\ &= \sum_{l} \frac{1}{l!} \sum_{i_1 \cdots i_l}^{\neq} \frac{1}{l} \sum_{j \neq k} [(x_{i_k}^{\beta} - x_{i_j}^{\beta})\, \partial_{i_k}^{\alpha} - (x_{i_k}^{\alpha} - x_{i_j}^{\alpha})\, \partial_{i_k}^{\beta}] \\ &\quad \times V^{(l)}(r_{i_1}, ..., r_{i_l}) \int_0^1 ds\, \delta(r - s r_{i_k} - (1-s)\, r_{i_j}) \\ &= \sum_{l} \frac{1}{l!} \sum_{i_1 \cdots i_l}^{\neq} \frac{1}{l} \sum_{j \neq k} [(x_{i_k}^{\beta} \partial_{i_k}^{\alpha} - x_{i_k}^{\alpha} \partial_{i_k}^{\beta}) - x_{i_j}^{\beta} \partial_{i_k}^{\alpha} + x_{i_j}^{\alpha} \partial_{i_k}^{\beta}] \\ &\quad \times V^{(l)}(r_{i_1}, ..., r_{i_l}) \int_0^1 ds\, \delta(r - s r_{i_k} - (1-s)\, r_{i_j})\end{aligned} \tag{2.11}$$

With the aid of (2.4) and (2.7), we convert this expression into

$$
\begin{aligned}
&\sigma^{\alpha\beta}(r)-\sigma^{\beta\alpha}(r)\\
&\quad=\sum_l \frac{1}{l!}\sum_{i_1\cdots i_l}^{\neq}\frac{1}{l}\sum_{j\neq k}\left[(x_{i_k}^{\beta}\partial_{i_k}^{\alpha}-x_{i_k}^{\alpha}\partial_{i_k}^{\beta})-x_{i_j}^{\beta}\partial_{i_k}^{\alpha}+x_{i_j}^{\alpha}\partial_{i_k}^{\beta}\right]\\
&\qquad\times V^{(l)}(r_{i_1},\dots,r_{i_l})\int_0^1 ds\,\left[\delta(r-sr_{i_k}-(1-s)\,r_{i_j})-\delta(r-r_{i_j})\right]\\
&\quad=\sum_l \frac{1}{l!}\sum_{i_1\cdots i_l}^{\neq}\frac{1}{l}\sum_{j\neq k}\left[(x_{i_k}^{\beta}-x_{i_j}^{\beta})\,\partial_{i_k}^{\alpha}-(x_{i_k}^{\alpha}-x_{i_j}^{\alpha})\,\partial_{i_k}^{\beta}\right]\\
&\qquad\times V^{(l)}(r_{i_1},\dots,r_{i_l})\\
&\qquad\times\int_0^1 ds\int_0^1 dt\,\frac{d}{dt}\,\delta(r-tsr_{i_k}-t(1-s)\,r_{i_j}-(1-t)\,r_{i_j})\\
&\quad=-\sum_l \frac{1}{l!}\sum_{i_1\cdots i_l}^{\neq}\frac{1}{l}\sum_{j\neq k}\left[(x_{i_k}^{\beta}-x_{i_j}^{\beta})\,\partial_{i_k}^{\alpha}-(x_{i_k}^{\alpha}-x_{i_j}^{\alpha})\,\partial_{i_k}^{\beta}\right]\\
&\qquad\times V^{(l)}(r_{i_1},\dots,r_{i_l})\\
&\qquad\times\int_0^1 ds\int_0^1 dt\sum_{\gamma}s(x_{i_k}^{\gamma}-x_{i_j}^{\gamma})\,\partial^{\gamma}\delta(r-tsr_{i_k}-(1-ts)\,r_{i_j})
\end{aligned}
\tag{2.12}
$$

Therefore, (2.8) is fulfilled with

$$
\begin{aligned}
\eta^{\alpha\beta\gamma}(r)=&-\sum_l \frac{1}{l!}\sum_{i_1\cdots i_l}^{\neq}\frac{1}{l}\sum_{j\neq k}(x_{i_k}^{\gamma}-x_{i_j}^{\gamma})\\
&\times\left[(x_{i_k}^{\beta}-x_{i_j}^{\beta})\,\partial_{i_k}^{\alpha}-(x_{i_k}^{\alpha}-x_{i_j}^{\alpha})\,\partial_{i_k}^{\beta}\right]V^{(l)}(r_{i_1},\dots,r_{i_l})\\
&\times\int_0^1 ds\int_0^s dt\,\delta(r-tr_{i_k}-(1-t)\,r_{i_j})
\end{aligned}
\tag{2.13}
$$

Moreover, as required, this expression has the property

$$
\eta^{\alpha\beta\gamma}(r)=\eta^{\alpha\beta\gamma}(r;\{r_i\})=\eta^{\alpha\beta\gamma}(r+a;\{r_i+a\})
\tag{2.14}
$$

Thus, the explicit construction and symmetrization of physically well-behaved stress tensor fields has been completely carried through.

Evidently, the above considerations have also supplied us with the proper expressions which generalize the results of refs. 3–5 on correlation functions involving stress tensor fields in the presence of spontaneously broken symmetries. In addition, I should mention that—due to (2.8)—there are two different forms for a continuity equation involving the angular momentum density, i.e.,

$$\frac{\partial}{\partial t} l^{\alpha\beta}(r) = \sum_{\gamma} \frac{\partial}{\partial x^{\gamma}} [x^{\alpha}\sigma_S^{\beta\gamma}(r) - x^{\beta}\sigma_S^{\alpha\gamma}(r)] \tag{2.15}$$

or

$$\frac{\partial}{\partial t} l^{\alpha\beta}(r) = \sum_{\gamma} \frac{\partial}{\partial x^{\gamma}} [x^{\alpha}\sigma^{\beta\gamma}(r) - x^{\beta}\sigma^{\alpha\gamma}(r) + \eta^{\alpha\beta\gamma}(r)] \tag{2.16}$$

respectively. This means in particular that there are two distinct ways to generalize the considerations of ref. 4 which deal with self-sustained orientational inhomogeneities. One has the choice to consider correlations involving $\sigma_S^{\alpha\beta}$ or such involving $\sigma^{\alpha\beta}$ and $\eta^{\alpha\beta\gamma}$.

3. THE QUANTUM CASE

I conclude the present paper with some remarks on the construction and symmetrization of a quantum mechanical stress tensor field. It is easy to see that the procedures of Section 2 can largely be carried over. Introducing the usual annihilation (creation) operators ψ (ψ^+) satisfying the (anti)commutation relations

$$\begin{gathered} \psi(r)\,\psi^+(r') \pm \psi^+(r')\,\psi(r) = \delta(r-r') \\ \psi^+(r)\,\psi^+(r') \pm \psi^+(r')\,\psi^+(r) = 0 = \psi(r)\,\psi(r') \pm \psi(r')\,\psi(r) \end{gathered} \tag{3.1}$$

one has

$$p^{\alpha}(r) = (\hbar/2i)[\psi^+(r)\,\partial^{\alpha}\psi(r) - \partial^{\alpha}\psi^+(r)\,\psi(r)] \tag{3.2}$$

and

$$\begin{aligned} H = \frac{\hbar^2}{2m} \int dr\, \nabla\psi^+(r)\,\nabla\psi(r) + \sum_l \frac{1}{l!} \int dr_1 \cdots dr_l\, V^{(l)}(r_1,..., r_l) \\ \times\, \psi^+(r_1)\cdots\psi^+(r_l)\,\psi(r_l)\cdots\psi(r_1) \end{aligned} \tag{3.3}$$

Therefore

$$\frac{\partial}{\partial t} p^{\alpha}(r) = \frac{[p^{\alpha}(r), H]}{i\hbar}$$

$$= -\frac{\hbar^2}{2m} \sum_{\beta} \partial^{\beta} \left\{ \partial^{\alpha}\psi^{+}(r)\, \partial^{\beta}\psi(r) + \partial^{\beta}\psi^{+}(r)\, \partial^{\alpha}\psi(r) \right.$$

$$\left. - \frac{1}{2} \partial^{\alpha}\partial^{\beta}[\psi^{+}(r)\, \psi(r)] \right\}$$

$$- \sum_{l} \frac{1}{l!} \int dr_1 \cdots dr_l \sum_{k=1}^{l} \partial_k^{\alpha} V^{(l)}(r_1,..., r_l)\, \delta(r - r_k)$$

$$\times \psi^{+}(r_1) \cdots \psi^{+}(r_l)\, \psi(r_l) \cdots \psi(r_1) \tag{3.4}$$

With the translation invariance condition (2.4) the interaction part is written as follows:

$$- \sum_{l} \frac{1}{l!} \int dr_1 \cdots dr_l \sum_{k=1}^{l} \partial_k^{\alpha} V^{(l)}(r_1,..., r_l)\, \delta(r - r_k)$$

$$\times \psi^{+}(r_1) \cdots \psi^{+}(r_l)\, \psi(r_l) \cdots \psi(r_1)$$

$$= -\sum_{l} \frac{1}{l!} \int dr_1 \cdots dr_l \frac{1}{l} \sum_{j,k=1}^{l} \partial_k^{\alpha} V^{(l)}(r_1,..., r_l)$$

$$\times [\delta(r - r_k) - \delta(r - r_j)]\, \psi^{+}(r_1) \cdots \psi^{+}(r_l)\, \psi(r_l) \cdots \psi(r_1)$$

$$= \sum_{l} \frac{1}{l!} \int dr_1 \cdots dr_l \frac{1}{l} \sum_{j \neq k} \partial_k^{\alpha} V^{(l)}(r_1,..., r_l) \int_0^1 ds \sum_{\beta} (x_k^{\beta} - x_j^{\beta})$$

$$\times \partial^{\beta}\delta(r - sr_k - (1 - s)\, r_j)\, \psi^{+}(r_1) \cdots \psi^{+}(r_l)\, \psi(r_l) \cdots \psi(r_1) \tag{3.5}$$

The stress tensor is immediately read off:

$$\sigma^{\alpha\beta}(r) = -\frac{\hbar^2}{2m} \left\{ \partial^{\alpha}\psi^{+}(r)\, \partial^{\beta}\psi(r) + \partial^{\beta}\psi^{+}(r)\, \partial^{\alpha}\psi(r) \right.$$

$$\left. - \frac{1}{2} \partial^{\alpha}\partial^{\beta}[\psi^{+}(r)\, \psi(r)] \right\}$$

$$+ \sum_{l} \frac{1}{l!} \int dr_1 \cdots dr_l \frac{1}{l} \sum_{j \neq k} (x_k^{\beta} - x_j^{\beta})\, \partial_k^{\alpha} V^{(l)}(r_1,..., r_l)$$

$$\times \int_0^1 ds\, \delta(r - sr_k - (1 - s)\, r_j)\, \psi^{+}(r_1) \cdots \psi^{+}(r_l)\, \psi(r_l) \cdots \psi(r_1) \tag{3.6}$$

This tensor field has a symmetric kinetic part and an in general nonsymmetric interaction part. This interaction term is an exact structural analog of the classical expression in (2.6). The whole symmetrization procedure of Section 2 therefore can also be applied to the quantum case. As the further computations and results are completely evident, it is not necessary to give them explicitly.

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